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A variant of the low-Reynolds-number two-equation turbulence model applied to variable property mixed convection flows

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Previous work has demonstrated that the low-Reynolds-number model of Launder and Sharma (1974) offers significant advantages over other two-equation turbulence models in the computation of highly non-universal buoyancy-influenced (or "mixed convection") pipe flows. It is known, however, that the Launder and Sharma model does not possess high quantitative accuracy in regard to simpler forced convection flows. A variant of the low-Reynolds-number scheme is developed here by reference to data for constant property forced convection flows. The re-optimized model and the Launder and Sharma formulation are then examined against experimental measurements for mixed convection flows, including cases in which variable property effects are significant.

Keywords: mixed convection; variable properties; vertical pipe flow; turbulence models

Introduction

The relative simplicity of two-equation turbulence models has resulted in their widespread application to problems of turbulent fluid mechanics and convective heat transfer. The models consist of a constitutive equation which relates Reynolds stress to mean strain rate via a turbulent viscosity and transport equations for the turbulent kinetic energy k and its rate of dissipation ϵ (although other parameters have been selected as the second scale-determining variable). In thin shear flows, where only one component of the Reynolds stress tensor is active in the mean flow equations, modelling at the two-equation level represents the least complex route by which one might expect to achieve accurate resolution of some "non-universal" flow features.

A class of thin shear flows which exhibit very marked departures from universal behaviour (such as adherence to the "law of the wall") are mixed convection pipe flows. The regime of mixed convection occurs in vertical pipe flows which are characterized by relatively low bulk Reynolds number and relatively high Grashof number. Thus, the "reference" forced flow (defined in terms of Reynolds and Prandtl numbers) is significantly modified by the action of buoyancy. In the ascending flow case (to which present attention is restricted), the effectiveness of heat transfer with respect to forced convection is at first reduced with increasing buoyancy influence; a further increase promotes a partial recovery, and, at high levels of buoyancy influence, there is eventual enhancement of heat transfer effectiveness. Reviews of mixed convection research are provided by Petukhov and Polyakov (1988) and Jackson et al. (1989).

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Two-equation models and their application to turbulent mixed convection

The standard "high-Reynolds-number" two-equation k-ϵ model is presented by Launder and Spalding (1974) and, in modelling terms, represents an asymptote towards which models incorporating low-Reynolds-number features revert in fully turbulent flow regions. The modification of the high-Reynolds-number form to include functions and additional terms dependent on local flow features reduces reliance on assumptions of universal behaviour. In particular, low-Reynolds-number closures are integrated up to a solid boundary and therefore do not require the specification of "wall functions" as used in high-Reynolds-number formulations to bridge the near-wall region. The first model to include a "damping function," $f_\mu(\mathrm{Re}_t)$, in the constitutive equation was developed by Jones and Launder (1972); a re-optimization of that model due to Launder and Sharma (1974) (hereafter LS) has since achieved some status as the benchmark low-Reynolds-number k-ϵ formulation.

The general strategy adopted in order to develop or refine turbulence closures is to specify model parameters (e.g., the turbulent Reynolds number) and then to "tune" model constants and functions by reference to "simple" flows such as fully developed channel or pipe flow. The level of activity in the development of alternative two-equation turbulence models is demonstrated by the review of Patel et al. (1985) in which 10 models were examined against data for boundary layers. It was found that only four models (including the LS model) performed at all satisfactorily and that weaknesses were apparent in all the models tested. Betts and Dafa'Alla (1986) tested 11 two-equation models (substantially the same models as those considered by Patel et al.) against data for natural convection cavity flow and found that only 4 models were in reasonable agreement with measurements. The LS formulation was the only model deemed to yield broadly satisfactory results by both Patel et al. and Betts and Dafa'Alla.

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Cotton and Jackson (1990) compared a constant properties (Boussinesq approximation) formulation of the LS model against the mixed convection air flow measurements of Carr et al. (1973) and Steiner (1971) and found generally acceptable agreement with the experimental data. That work has recently been extended by Cotton et al. (1995) using the LS model in variable property form and including comparison with the data of Polyakov and Shindin (1988). Other recent comparative studies by the authors and their colleagues (Cotton and Kirwin 1993; Mikielewicz 1994; Kirwin 1995) have demonstrated that, of the models tested, the LS model produces the most accurate results in the computation of ascending mixed convection air flows.

Development of a variant of the low-Reynolds-number k-ϵ model

Guiding considerations

The principal objective of the present research is to probe further the reasons for the success of the LS model in the calculation of mixed convection flows. An essential element of that model is that all departures from the high-Reynolds-number format are specified in terms of local flow variables. Thus, we address the question: "Is it possible to reformulate the LS model to achieve better agreement with simple flows and subsequently retain a comparable level of performance in mixed convection computations?" In other terms, we seek to investigate sensitivity to model detail while retaining the same model parameters.

Model equations

The equations of the LS model and the present variant may be written in the following generic form:

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\rho k \tag{1}$$

$$\mu_t = C_\mu f_\mu(\mathrm{Re}_t)\frac{\rho k^2}{\tilde{\epsilon}} \tag{2}$$

$$\frac{D(\rho k)}{Dt} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] - \rho(\tilde{\epsilon} + D_\epsilon) \tag{3}$$

$$\frac{D(\rho\tilde{\epsilon})}{Dt} = C_{\epsilon 1}\frac{\tilde{\epsilon}}{k}\mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\frac{\partial \tilde{\epsilon}}{\partial x_j}\right] - C_{\epsilon 2}f_2\frac{\rho\tilde{\epsilon}^2}{k} + C_{\epsilon 3}\frac{\mu\mu_t}{\rho}\left(\frac{\partial^2 U_j}{\partial x_k \partial x_l}\right)^2 \tag{4}$$

The high-Reynolds-number constants in both models are unadjusted from those quoted by Launder and Spalding (1974):

$$C_\mu = 0.09\,; \quad \sigma_k = 1.0\,; \quad \sigma_\epsilon = 1.3\,;$$
$$C_{\epsilon 1} = 1.44\,; \quad C_{\epsilon 2} = 1.92 \tag{5}$$

Notation

Bo	buoyancy parameter, Equation 12
c_f	local friction coefficient, $\tau_w/\frac{1}{2}\rho U_b^2$
c_p	specific heat capacity at constant pressure
$C_{\epsilon 1}$, $C_{\epsilon 2}$, $C_{\epsilon 3}$	constants in production and sink terms of ϵ-equation
C_μ	constant in constitutive equation of k-ϵ model
D	pipe internal diameter
D_ϵ	dissipation term in k-equation ($\epsilon = \tilde{\epsilon} + D_\epsilon$)
f_2	function in sink term of ϵ-equation
f_μ	function in constitutive equation of k-ϵ model
g	magnitude of acceleration due to gravity
Gr	Grashof number based on wall heat flux, $\beta_b g q D^4/\lambda_b \nu_b^2$
H	channel half-width
k	turbulent kinetic energy
k^+	k/U_τ^2
Nu	Nusselt number, $qD/\lambda_b(\theta_w - \theta_b)$
Nu_0	Nusselt number for forced convection
Pr	Prandtl number, $c_p\mu_b/\lambda_b$
q	wall heat flux
q^+	heat loading parameter, Equation 13
Re	Reynolds number for pipe flow, $\rho_b U_b D/\mu_b$
Re_H, Re_τ	Reynolds numbers for channel flow, $(2\rho U_b H/\mu)$, $(\rho U_\tau H/\mu)$
Re_t	turbulent Reynolds number, $k^2/\nu\tilde{\epsilon}$
$-\overline{uv}$	Reynolds stress
$-\overline{uv}^+$	$-\overline{uv}/U_\tau^2$
U	streamwise velocity
U^+	U/U_τ
U_i, u_i	mean, fluctuating velocity components in Cartesian tensors
U_τ	friction velocity, $\sqrt{\tau_w/\rho}$
x	axial coordinate
x_i	Cartesian coordinates
y	normal distance from wall
y^+	yU_τ/ν
Greek	
β	coefficient of volume expansion
δ_{ij}	Kronecker delta
ϵ	rate of dissipation of k
ϵ^+	$\epsilon\nu/U_\tau^4$
$\tilde{\epsilon}$	modified dissipation variable
θ	temperature
λ	thermal conductivity
μ	dynamic viscosity
ν	kinematic viscosity, μ/ρ
ρ	density
σ_k, σ_ϵ	turbulent Prandtl number for diffusion of k, ϵ
τ	shear stress
Subscripts	
b	bulk
t	turbulent
w	wall
Superscript	
$-$	mean

Table 1 Low-Reynolds-number model features

Model	f_μ	f_2	$C_{\epsilon 3}$
Launder and Sharma 1974	$\exp\left[\dfrac{-3.4}{(1+\mathrm{Re}_t/50)^2}\right]$	$1-0.3\exp[-\mathrm{Re}_t^2]$	2.0
Present formulation	$1-0.97\exp[-\mathrm{Re}_t/160]$ $-0.0045\,\mathrm{Re}_t\exp[-(\mathrm{Re}_t/200)^3]$	1.0	0.95

In both models D_ϵ (where $\tilde{\epsilon}=\epsilon-D_\epsilon$) is defined here as follows:

$$D_\epsilon = 2\nu\left(\partial k^{1/2}/\partial x_j\right)^2 \quad \text{for} \quad y^+ \geqslant 2 \tag{6a}$$

$$D_\epsilon = 2\nu k/x_j^2 \quad \text{for} \quad y^+ < 2 \tag{6b}$$

Ismael (1993) has demonstrated that neither Equation 6a nor 6b is consistent with a quadratic variation of k in the immediate near-wall region when solution is obtained using a linear discretization scheme (such as employed here). It is found, however, that the "switch" applied using Equation 6b in the region $y^+<2$ promotes convergence of the discretized equation set. The models differ in terms of the low-Reynolds-number functions f_μ and f_2 and the constant $C_{\epsilon 3}$ appearing in the final term of Equation 4. Table 1 details these elements of closure as adopted in the LS model and the present development.

Results of a forced convection optimization exercise

The closure elements f_μ and $C_{\epsilon 3}$ have been optimized by tuning the present model to data for forced convection channel and pipe flows. (The function $f_2(\mathrm{Re}_t)$ which appears in the dissipation equation of the LS model departs from unity only at very low values of turbulent Reynolds number and has been omitted in the present exercise.) It should be noted that the model is tested against mixed convection flows below without having made any reference to such flows in the optimization procedure.

The mean flow equations of continuity, momentum, and energy are solved in conjunction with either the present or LS turbulence model. Turbulent Prandtl number for diffusion of heat is set to 0.9 throughout this study. The governing equations are written in constant property thin shear form and the solution is marched in the flow direction until a fully developed condition is attained. A finite volume/finite difference discretization scheme is employed following Leschziner (1982). The near-wall node is positioned at $y^+\simeq 0.5$ and 100 control volumes span the pipe radius (or channel half-width). Two algorithms are used to generate a grid distribution that expands from the wall; 50 control volumes are located in the region $y^+\lesssim 60$. A range of sensitivity tests was performed to ensure that results are sensibly independent of details of the numerical procedure. In terms of numerical stability and computing times, no significant difference was noted between the present and LS models.

Comparison is made below with correlations for flow resistance and heat transfer in pipe flow and with Direct Numerical Simulation (DNS) data for channel flow profiles.

Local friction coefficient and Nusselt number for pipe flow

Results are reported for Reynolds numbers of 5×10^3, 10^4, 5×10^4 and 10^5, and Prandtl numbers of 0.7 and 6.95. The thermal boundary condition of uniform wall heat flux is applied. Calculated local friction coefficient is compared against the Prandtl equation (Schlichting 1979):

$$c_f = \left(4\log_{10}\left(2\,\mathrm{Re}\,c_f^{0.5}\right)-1.6\right)^{-2} \tag{7}$$

Values of computed Nusselt number are compared against the Petukhov and Kirillov correlation (Kays and Crawford 1980):

$$\mathrm{Nu}_0 = \frac{(c_f/2)\,\mathrm{Re}\,\mathrm{Pr}}{1.07+12.7(c_f/2)^{0.5}(\mathrm{Pr}^{2/3}-1)} \tag{8}$$

where $c_f = 0.25\,(1.82\log_{10}\mathrm{Re}-1.64)^{-2}$.

In the case Pr = 0.7, comparison is also made with a form of the Dittus–Boelter equation proposed by Kays and Leung (1963):

$$\mathrm{Nu}_0 = 0.022\,\mathrm{Re}^{0.8}\,\mathrm{Pr}^{0.5} \tag{9}$$

Schlichting (1979) notes that Equation 7 is in good agreement with data over a wide range of Reynolds number. Equation 8 holds for $10^4 \leqslant \mathrm{Re} \leqslant 5\times10^6$ and moderate to high Prandtl

Table 2 Local friction coefficient and Nusselt number

		Re			
		5×10^3	10^4	5×10^4	10^5
$c_f\times10^3$:	Equation 7	9.35	7.72	5.22	4.50
	Launder and Sharma 1974 model	8.86	7.32	5.04	4.38
	Present model	9.56	7.79	5.29	4.58
$\mathrm{Nu}_0(\mathrm{Pr}=0.7)$:	Equation 8	—	30.5	98.2	166.8
	Equation 9	—	29.2	105.7	184.1
	LS model	16.9	29.1	103.8	181.1
	Present model	18.2	31.1	109.2	189.5
$\mathrm{Nu}_0(\mathrm{Pr}=6.95)$:	Equation 8	—	86.1	326.3	586.8
	LS model	35.9	68.6	288.9	533.2
	Present model	45.9	84.6	343.2	627.5

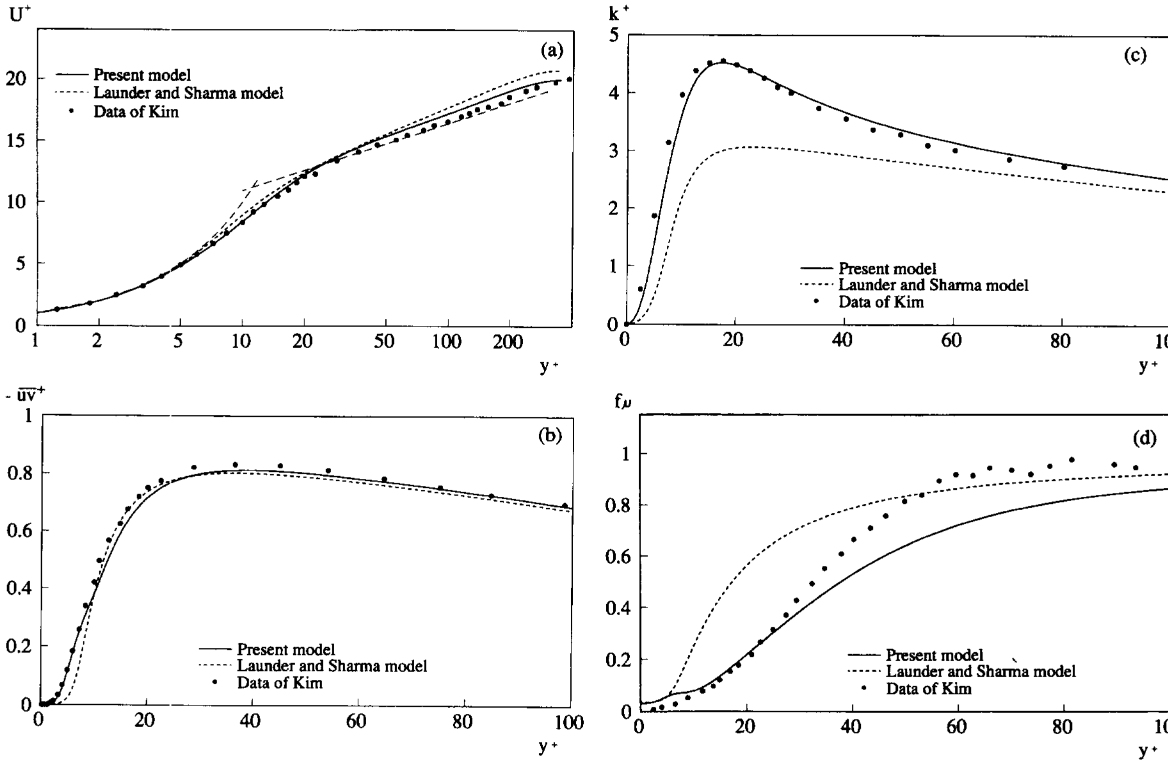

Figure 1 Channel flow profiles at $Re_\tau = 395$; (a) velocity, (b) Reynolds stress, (c) turbulent kinetic energy, and (d) damping function

numbers to within approximately $\pm 10\%$ (Kays and Crawford 1980). Kays and Leung (1963) report that Equation 9 has been used to correlate a large amount of data for gas flows and demonstrate good agreement with data for $10^4 \leqslant \mathrm{Re} \leqslant 10^5$ and $\mathrm{Pr} = 0.7$.

Local friction coefficient and Nusselt number computed using the two turbulence models are compared against Equations 7–9 in Table 2. Over the range $5 \times 10^3 \leqslant \mathrm{Re} \leqslant 10^5$, values of c_f returned by the LS model lie between 3 and 5% below the Prandtl equation; the present model yields values of c_f between 1% and 2% above the Prandtl equation. In the calculation of heat transfer for air flows ($\mathrm{Pr} = 0.7$), the LS model yields Nusselt number discrepancies (for $10^4 \leqslant \mathrm{Re} \leqslant 10^5$) in the range -5% to $+9\%$ with respect to the Petukhov and Kirillov correlation; corresponding figures for the present model are $+2$ to $+14\%$. Comparison with the Kays and Leung (1963) correlation over the same Reynolds number range reveals that the maximum discrepancy returned by the LS model is -2%; the present model exhibits a maximum discrepancy of $+7\%$. More striking discrepancies are evident for water flows ($\mathrm{Pr} = 6.95$, and $10^4 \leqslant \mathrm{Re} \leqslant 10^5$): the LS model computes Nusselt numbers between 9% and 20% below Equation 8, whereas the present model returns values between 2% below and 7% above the correlation.

Velocity and turbulence profiles for channel flow

Comparison is made with the DNS data of Kim for $Re_\tau = 395$ (communicated to Michelassi et al. 1991). The bulk Reynolds number of the flow is $Re_H = 12{,}300$.

Figures 1a–1d show the present and LS models compared against the DNS data of Kim. Velocity profiles are shown in Fig. 1a together with the viscous sublayer profile and the 'law of the wall' with constants as specified by Patel and Head (1969):

$$U^+ = y^+ \tag{10}$$

$$U^+ = 5.45 + 5.5 \log_{10} y^+ \tag{11}$$

Distributions of Reynolds stress (Figure 1b) show that the present model is in better agreement with data in the "near-wall" ($y^+ \lesssim 10$) and "logarithmic" regions ($y^+ \gtrsim 30$), whereas the LS model is closer to data over the intermediate "buffer" region. The profiles of turbulent kinetic energy (Figure 1c) show the present model to be in considerably better agreement with data. Results similar to those obtained above for U^+ and $-\overline{uv}^+$ are generated at $Re_\tau = 180$ (not shown); k^+ is, however, inadequately resolved by the present model at values of y^+ beyond the maximum in k^+ (Kirwin 1995).

Figure 1d shows the distribution of the damping function f_μ at $Re_\tau = 395$. The present model is significantly closer to the DNS data near the wall ($y^+ \leqslant 20$), whereas the LS model is in better agreement with data in the region $y^+ \geqslant 60$. Neither model could be said overall to be in satisfactory agreement with the data.

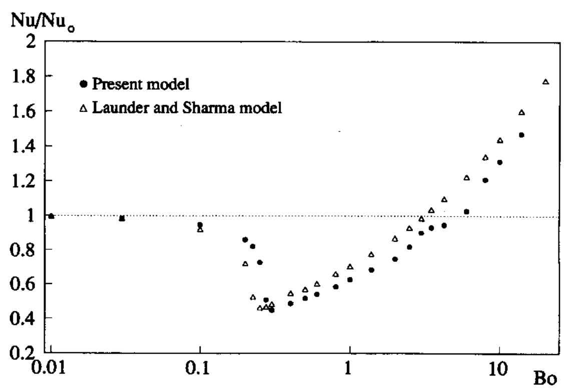

Figure 2 Mixed convection Nusselt number normalized by the forced convection value as a function of buoyancy parameter

Table 3 Conditions of the experiments of Steiner 1971 and Vilemas et al. 1992

Source	Figure	Re	$Gr \times 10^{-7}$	Bo	$q^+ \times 10^4$
Steiner 1971	3a	14900	21.7	0.12	10.2
	3b	9800	12.2	0.28	8.7
	3c	7100	11.7	0.80	11.6
	3d	5000	9.30	2.1	13.0
Vilemas et al. 1992	4a	19400	45.5	0.10	4.0
	4b	13300	24.1	0.19	3.0
	4c	8850	18.5	0.60	3.3
	4d	3700	11.0	7.0	4.5

Results for turbulent mixed convection

The mean flow and turbulence model equations are now written in full variable properties form (details are given by Kirwin 1995). Direct buoyant production terms in the k- and ϵ-equations are omitted following the constant properties study of Cotton and Jackson (1990) who reported the terms to exert, at most, a second-order influence. The present authors consider, however, that in light of the present more refined modelling framework, this approximation warrants re-examination in any future work. Numerical procedures are essentially the same as those employed earlier to obtain fully developed profiles in constant properties forced convection, however a small axial step (of approximately two viscous sublayer widths) is now employed to ensure accurate resolution of rapid development effects. The thermal boundary condition of uniform wall heat flux is applied in all cases.

Figure 2 gives an overview of the modifications to heat transfer effectiveness occurring in ascending turbulent mixed convection flows. Computations for this figure (only) are for constant property formulations of the governing equations within the Boussinesq approximation. The extent of buoyancy influence is characterized in terms of a buoyancy parameter developed originally by Hall and Jackson (1969) and formulated here in the form quoted by Jackson et al. (1989):

$$\mathrm{Bo} = 8 \times 10^4 \frac{\mathrm{Gr}}{\mathrm{Re}^{3.425}\,\mathrm{Pr}^{0.8}} \tag{12}$$

Mixed convection Nusselt number normalized by the forced convection value at the same Reynolds and Prandtl numbers (in this case, $\mathrm{Re} = 5{,}000$, $\mathrm{Pr} = 0.7$, and $x/D = 100$) is seen to follow the pattern of impairment-followed-by-recovery described in the Introduction. Calculations using the present and LS models show similar patterns of behaviour, but differ quantitatively in terms of the computed levels of impairment or enhancement of heat transfer effectiveness. The irregularities apparent in the points obtained using the present model are due to the occurrence of local recoveries in Nusselt number at the particular axial location selected.

The remaining figures presented here show case-by-case thermal development in ascending mixed convection air flows for the conditions of experiments reported by Steiner (1971) and Vilemas et al. (1992). The two sets of data encompass a range of buoyancy influence and variable property effects, the latter being quantified in terms of the "heat loading parameter" q^+:

$$q^+ = \frac{q}{\rho_b U_b c_p \theta_b} \tag{13}$$

Table 3 shows the values of Reynolds number, Grashof number, Bo, and q^+ prevailing at the start of heating for the eight cases examined here. The Prandtl number is approximately equal to 0.7 at $x/D = 0$ in all cases. In considering the data of Steiner (1971), published values of Nusselt number are multiplied

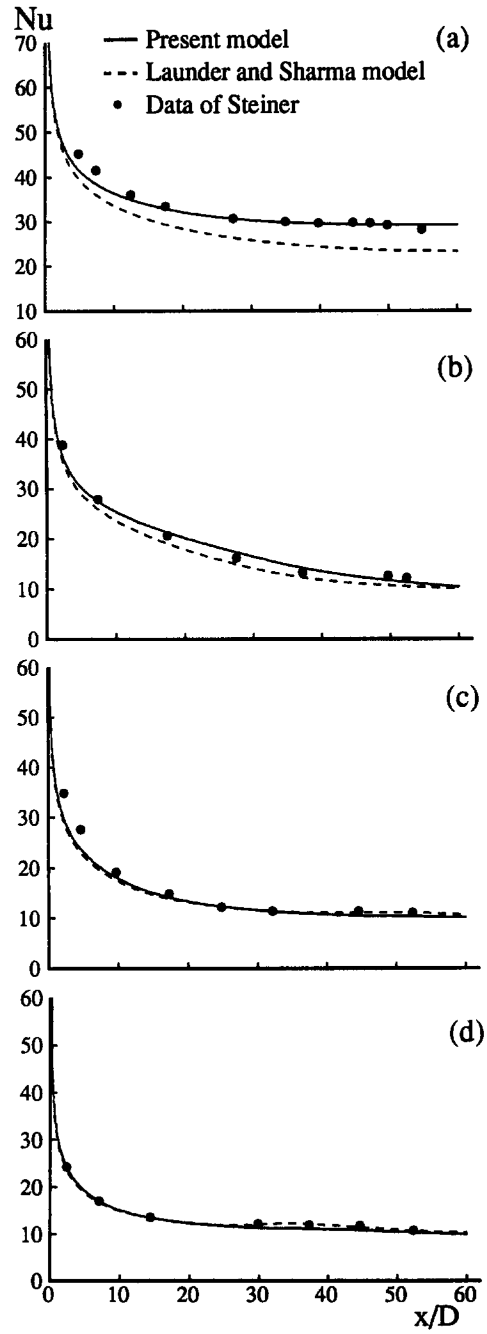

Figure 3 Nusselt number development for the conditions of Steiner's experiments (conditions as given in Table 3)

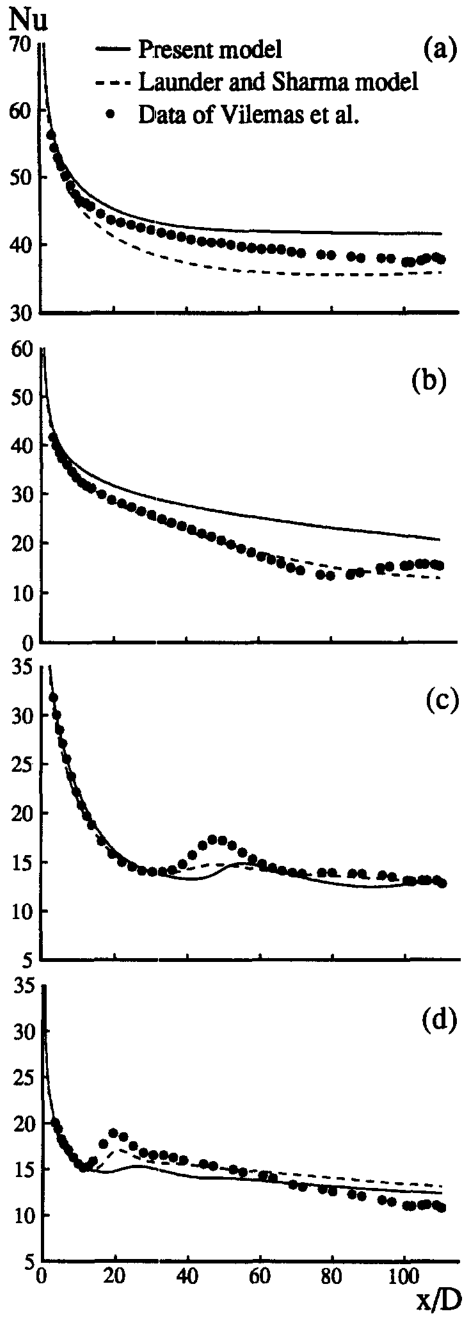

Figure 4 Nusselt number development for the conditions of the Vilemas et al. (1992) experiments (conditions as given in Table 3)

by $\lambda_{x=0}/\lambda_{local}$ to obtain experimental distributions of Nu based on local fluid properties.

Figure 3 shows Steiner's (1971) data compared with the present and LS models. The level of heat loading is roughly uniform and the buoyancy parameter increases from approximately 0.1 to 2.0 (Table 3). The case of lowest Bo (Figure 3a) shows the present model to be in better agreement with data than the LS model. An increase in buoyancy parameter through the region of maximum impairment and into the recovery region (Figures 3b–d) reveals both models to be in good agreement with data.

The recent data of Vilemas et al. (1992) shown in Figure 4 represent closely spaced measurements of Nusselt number obtained using a test section of over 100 diameters in length. (The data used here are taken from detailed documentation supplied by Poškas which supplements the results presented by Vilemas et al. A selection of results is presented; Kirwin 1995 reports a comprehensive series of comparisons with the data.) Figures 4a–d show data and model computations for relatively low, approximately constant, values of q^+; Bo is increased from 0.1 to 7.0 over the four cases. It is seen from Figure 4a that the present model returns values of Nusselt number above the data, whereas the LS model produces lower values. Figure 4b is the clearest example in the present study of a case where the LS model performs considerably better then the present formulation. (The buoyancy parameter of this case represents the maximum impairment region and it should be noted that heat transfer performance is highly sensitive to the precise conditions of an experiment, cf. Figure 2.) At higher levels of buoyancy influence the experimental data of Figures 4c and d clearly show local recoveries in Nusselt number. The LS model more accurately captures the locations of these maxima, however neither model exhibits high quantitative accuracy.

Concluding remarks

In the work reported here a variant of the Launder and Sharma (1974) low-Reynolds-number k-ϵ turbulence model is developed on the basis of an empirical optimization procedure. The present model is generally seen to have been tuned to be in better agreement with the reference forced convection cases (which include DNS data produced some time after the publication of the LS model). Comparison of the two models with experimental data for mixed convection flows shows that, in some cases, the performance of the LS model is superior to that of the present variant, whereas, in other cases, the reverse is true. Thus, no major improvement of a standard closure could be said to have been developed. It is demonstrated, however, that results for mixed convention *comparable* to those obtained using the LS model can be generated by an alternative Re_t-damped model.

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